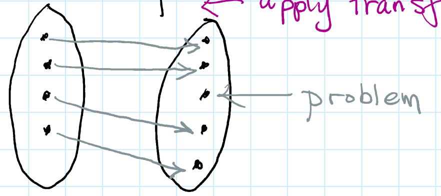
10-Feb

Monday, February 10, 2025 1:36 PM

transformations (cont'd)

Like functions, not every matrix has an inverse.

T ← apply transformation (function)



X domain $\qquad$ Y target space

ex. $y = (x_1)^2 + (x_2)^2 + (x_3)^2$ $\qquad$ input: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\qquad$ output: y (scalar)

$$\mathbb{R}^3 \longrightarrow \mathbb{R}$$

ex $\vec{r} = \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix}$ $\qquad$ input: t $\qquad$ output: $\vec{r}$

$$\mathbb{R} \longrightarrow \mathbb{R}^3$$

<u>Linear Transformation</u> := a function T from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $\exists$ $n \times m$ matrix s.t. $T(\vec{x}) = A\vec{x}$

there exists $\qquad$ where $A\vec{x}$ is in vector space $\mathbb{R}^m$

generalize: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \vec{x}$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \cdot x_1 - 1 \cdot x_2 \\ x_1 + 0x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

lengths of vectors are the same:
show using distance formula:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{vs.} \quad \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$$

$$L = \sqrt{(x_1)^2 + (x_2)^2} \qquad L = \sqrt{(-x_2)^2 + (x_1)^2}$$

Use <u>dot product</u> to show that the vectors are perpendicular:

$$\vec{x} \cdot T(\vec{x})$$

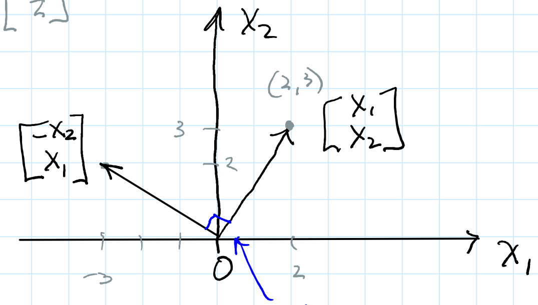
$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} = x_1(-x_2) + x_2(x_1)$$

$$-x_1 x_2 + x_1 x_2 = 0$$

ex. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$$-x_1x_2 + x_1x_2 = 0$$

ex. $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$



shows geometrically that $\vec{x}$ and $T(\vec{x})$ are perpendicular

ex. given $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

find $T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(0) + 3(0) \\ 4(1) + 5(0) + 6(0) \\ 7(1) + 8(0) + 9(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$

this returns column vector 1

can streamline the process:

find $T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$

2nd column vector

Do: find $T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}$

<u>Theorem</u>: given linear transformation, T, from $\mathbb{R}^m$ to $\mathbb{R}^n$, then the matrix of T is:

$$A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow & & \uparrow \\ T(\vec{e}_1) & T(\vec{e}_2) & \dots & T(\vec{e}_i) & \dots & T(\vec{e}_m) \\ \downarrow & \downarrow & & \downarrow & & \downarrow \end{bmatrix} \quad \text{where } \vec{e}_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow \text{ith component}$$

m components

show why:

given $A = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}$ then $T(\vec{e}_i) = A\vec{e}_i$

$$= \underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_m \end{bmatrix}}_{A} \overbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}}^{\vec{e}_i} \leftarrow \text{ith} = \vec{v}_i$$

the set of vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ in vector space $\mathbb{R}^m$

the set of vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_m$ in vector space $\mathbb{R}^m$ are called the <u>standard vectors</u>

in particular, for $\mathbb{R}^3$: $\vec{e}_1, \vec{e}_2, \vec{e}_3$ labeled $\vec{i}, \vec{j}, \vec{k}$

recall from § 1.3 that $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$ and

$A(k\vec{x}) = k(A\vec{x})$

where A is an $n \times m$ matrix,

$\vec{x}, \vec{y} \in \mathbb{R}^m$

↑ are elements of

k is a scalar

then $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$ and

$T(k\vec{v}) = k(T\vec{v})$ where $T(\vec{x}) = A\vec{x}$ from $\mathbb{R}^m$ to $\mathbb{R}^n$

ex. given the linear transformation

$$\begin{aligned} y_1 &= 7x_1 + 3x_2 - 9x_3 + 8x_4 \\ y_2 &= 6x_1 + 2x_2 - 8x_3 + 7x_4 \\ y_3 &= 8x_1 + 4x_2 + 7x_4 \end{aligned}$$

this is a function from $\mathbb{R}^4$ to $\mathbb{R}^3$

and it's represented by $A = \begin{bmatrix} 7 & 3 & -9 & 8 \\ 6 & 2 & -8 & 7 \\ 8 & 4 & 0 & 7 \end{bmatrix}$

↑ coefficient matrix

coefficient matrix of identity transformation $\mathbb{R}^n \to \mathbb{R}^n$ is

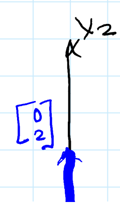
$n \times n$ <u>identity matrix</u>, I_n

ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ex. consider vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$, show how linear transformation

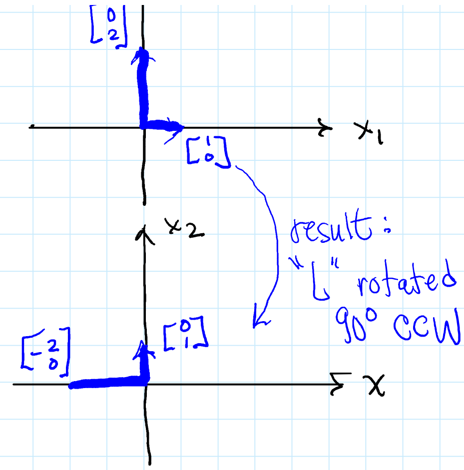
$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\vec{x}$ affects the vectors

$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

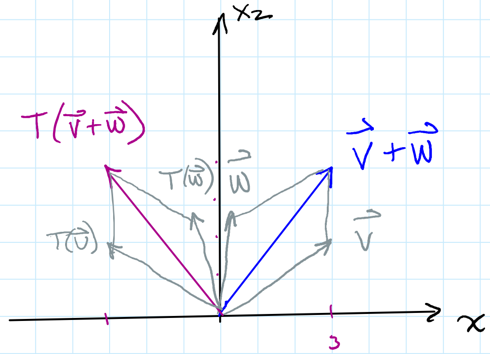


$$T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$$

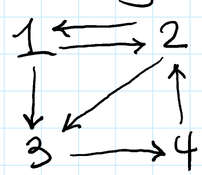


ex. given 90° CCW rotation $T(\vec{x}) = \begin{bmatrix}-x_2\\x_1\end{bmatrix}$



conclusion: addition property holds through transformation

ex. consider 4 URLs that may be viewed sequentially in a given pattern:



let x_1 x_2 x_3 x_4 be the proportions of the users that land on each of the pages initially – express as a <u>distribution</u> vector

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$$

for example: $\vec{x} = \begin{bmatrix}0.4\\0.1\\0.3\\0.2\end{bmatrix}$ or $\begin{bmatrix}2/5\\1/10\\3/10\\1/5\end{bmatrix}$

40% of users land here (page 1) first

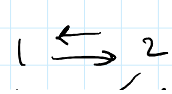
add exactly to 1 (100%)

users will move from page to page according to system

let's write result as transformation: $\vec{y} = \begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}$

where y_1 = user <u>ends</u> at page 1

y_2 = " 2

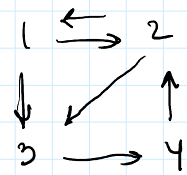


y_1 1

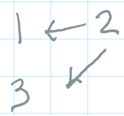
y_2 =   "   2

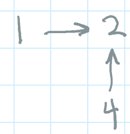
y_3 =   "   3

y_4 =   "   4



assume equal probability that user moves from one page to the next





$$y_1 = \tfrac{1}{2}x_2$$

$$y_2 = \tfrac{1}{2}x_1 \qquad\qquad 1x_4$$

$$y_3 = \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2$$

$$y_4 = x_3$$

check (add): $1x_1$ $1x_2$ $1x_3$ $1x_4$ ← all coefficients should add to 1 (account for all prob. of movements)

then matrix A:

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$